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The Secant Method for Nondifferentiable Operators

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Abstract—In this paper, we use the Secant method to find a solution of a nonlinear operator equation in Banach spaces. A semilocal convergence result is obtained. For that, we consider a condition for divided differences which generalizes the usual ones, i.e., Lipschitz continuous or Hölder continuous conditions. Besides, we apply our results to approximate the solution of a nonlinear equation. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The Secant method for solving a nonlinear equation in Banach spaces is a well-known iterative process [1]. An important feature of this method is that it does not use derivatives when it is applied. Let X, Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear operator and we consider the equation

$$F(x) = 0. \tag{1}$$

Let us denote by $\mathcal{L}(X,Y)$ the space of bounded linear operators from X to Y. An operator $[x, y; F] \in \mathcal{L}(X,Y)$ is called a divided difference of first order for the operator F on the points x and y $(x \neq y)$ if the following equality holds:

$$[x, y; F](x - y) = F(x) - F(y). \tag{2}$$

Using this definition, the Secant method is described by the following algorithm:

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n), \qquad x_0, x_{-1} \text{ given.} \tag{3}$$

The convergence of (3) to a solution of (1) has been studied by other authors [1–6]. The basic assumption is that the divided difference of first order for the operator F is Lipschitz or Hölder

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continuous in some ball around the initial iterate. But these assumptions force to operator F is differentiable [1,4]. In this work, we relax this requirement and just assume the following condition:

$$\|[x,y;F]-[v,w;F]\| \le \omega(\|x-v\|,\|y-w\|); \qquad x,y,v,w \in \Omega,$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing funtion in its two arguments. It is evident that this condition generalizes the conditions previously indicaded, by only considering $\omega(u_1,u_2) = k(u_1+u_2)$ for the Lipschitz continuous case and $\omega(u_1,u_2) = k(u_1^p+u_2^p)$ for the (k,p)-Hölder continuous one. Moreover, in general, this condition does not involve F is differentiable. Then we provide a semilocal convergence result for nondifferentiable operators in general. Finally, we give an example where the last is applied.

2. CONVERGENCE STUDY

THEOREM 2.1. *Assume that, for every pair of distinct points $x,y \in \Omega$, there exists a first-order divided difference $[x,y;F] \in \mathcal{L}(X,Y)$. Let $x_0, x_{-1} \in \Omega$ and assume*

(a) *the linear operator $L_0 = [x_{-1},x_0;F]$ is invertible and*

$$\left\|[x_{-1},x_0;F]^{-1}\right\| \le \beta, \qquad \|x_0-x_{-1}\| = \alpha, \qquad \left\|{L_0}^{-1}F(x_0)\right\| \le \eta;$$

(b) $\|[x,y;F]-[v,w;F]\| \le \omega(\|x-v\|,\|y-w\|)$; $x,y,v,w \in \Omega$, *where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing funtion in its two arguments;*

(c) *we denote by $m = \max\{\beta\omega(\alpha,\eta), \beta\omega(\eta,\eta)\}$ and assume that the equation*

$$u\left(1-\frac{m}{1-\beta\omega(u+\alpha,u)}\right)-\eta = 0 \tag{4}$$

has at least one positive zero, let R be the minimum positive one.

If $\beta\omega(R+\alpha,R)<1$, $M = m/(1-\beta\omega(R+\alpha,R)) < 1$, and $\overline{B(x_0,R)} \subset \Omega$, then the sequence $\{x_n\}$ given by (3) is well defined, remains in $\overline{B(x_0,R)}$ and converges to a unique solution x^* of equation (1) in $\overline{B(x_0,R)}$.

To simplify the notation, we denote $[x_{n-1},x_n;F] = L_n$. First, we prove, by mathematical induction, that the sequence given in (3) is well defined, namely iterative procedure (3) makes sense if, at each step, the operator $[x_{n-1},x_n;F]$ is invertible and the point x_{n+1} lies in Ω. From the initial hypotheses, it follows that x_1 is well defined and $\|x_1-x_0\| \le \eta < R$. Therefore, $x_1 \in B(x_0,R) \subseteq \Omega$.

Using (b) and assuming that ω is nondecreasing, we obtain

$$\begin{aligned}\left\|I-{L_0}^{-1}L_1\right\| &\le \left\|{L_0}^{-1}\right\| \, \|L_0-L_1\| \le \left\|{L_0}^{-1}\right\| \omega(\|x_{-1}-x_0\|,\|x_0-x_1\|)\\ &\le \beta\omega(\alpha,\eta) \le \beta\omega(R+\alpha,R) < 1,\end{aligned}$$

and, by the Banach lemma, ${L_1}^{-1}$ exists and

$$\left\|{L_1}^{-1}\right\| \le \frac{\beta}{1-\beta\omega(R+\alpha,R)},$$

and consequently, the iterate x_2 is well defined. Moreover, by (2) and (3), we get

$$F(x_1) = F(x_0) - [x_0,x_1;F](x_0-x_1) = (L_0-L_1)(x_0-x_1).$$

Then, by (b), we have

$$\|F(x_1)\| \le \|L_1-L_0\| \, \|x_1-x_0\| \le \omega(\|x_0-x_{-1}\|,\ \|x_1-x_0\|)\|x_1-x_0\| \le \omega(\alpha,\eta)\|x_1-x_0\|.$$

So, we obtain

$$\|x_2 - x_1\| \leq \|L_1{}^{-1}\| \ \|F(x_1)\| \leq \frac{m}{1-\beta\omega(R+\alpha,R)} \|x_1 - x_0\| = M\|x_1 - x_0\| < \eta.$$

On the other hand, if we take into account that R is a solution of (4), then

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (M+1)\|x_1 - x_0\| \leq (M+1)\eta < R,$$

and $x_2 \in B(x_0, R) \subseteq \Omega$.

Then, by induction, the following items are shown for $j \geq 1$:

(I_j) $\exists\, L_j{}^{-1} = [x_{j-1}, x_j; F]^{-1}$ such that $\|L_j{}^{-1}\| \leq \dfrac{\beta}{1-\beta\omega(R+\alpha,R)}$;

(II_j) $\|x_{j+1} - x_j\| \leq M\|x_j - x_{j-1}\| \leq M^j \|x_1 - x_0\| < \eta$.

Assuming that the linear operators L_j are invertible and $x_{j+1} \in B(x_0, R) \subseteq \Omega$ for all $j = 1, \dots, n-1$, we obtain

$$\begin{aligned}\|I - L_0{}^{-1}L_n\| &\leq \|L_0{}^{-1}\| \ \|L_0 - L_n\| \leq \beta\omega(\|x_{n-1} - x_{-1}\|, \|x_n - x_0\|)\\ &\leq \beta\omega(\|x_{n-1} - x_0\| + \|x_0 - x_{-1}\|, \|x_n - x_0\|) \leq \beta\omega(R+\alpha, R) < 1,\end{aligned}$$

and therefore,

$$\|L_n{}^{-1}\| \leq \frac{\beta}{1-\beta\omega(R+\alpha,R)}.$$

From the definition of the first divided difference and the Secant method, we can obtain

$$F(x_n) = F(x_{n-1}) - [x_{n-1}, x_n; F](x_{n-1} - x_n) = (L_{n-1} - L_n)(x_{n-1} - x_n).$$

Taking norms in the above equality and (b), we obtain

$$\begin{aligned}\|F(x_n)\| &\leq \|L_n - L_{n-1}\| \ \|x_n - x_{n-1}\| \leq \omega(\|x_{n-1} - x_{n-2}\|, \|x_n - x_{n-1}\|)\|x_n - x_{n-1}\|\\ &\leq \omega(\eta,\eta)\|x_n - x_{n-1}\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|L_n^{-1}\| \, \|F(x_n)\| \leq \frac{m}{1-\beta\omega(R+\alpha,R)} \|x_n - x_{n-1}\|\\ &= M\|x_n - x_{n-1}\| \leq M^n \|x_1 - x_0\| < \eta.\end{aligned}$$

Consequently, from (4) and (II_j), it follows:

$$\begin{aligned}\|x_{n+1} - x_0\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_{n-1}\| + \cdots + \|x_2 - x_1\| + \|x_1 - x_0\|\\ &\leq \left[M^n + M^{n-1} + \cdots + 1\right] \|x_1 - x_0\| \leq \left[\frac{1-M^{n+1}}{1-M}\right] \|x_1 - x_0\| < \frac{1}{1-M}\eta = R.\end{aligned}$$

So, $x_{n+1} \in B(x_0, R) \subseteq \Omega$ and the induction is complete.

Second, we prove that $\{x_n\}$ is a Cauchy sequence. For $k \geq 1$, we obtain

$$\begin{aligned}\|x_{n+k} - x_n\| &\leq \|x_{n+k} - x_{n+k-1}\| + \|x_{n+k-1} - x_{n+k-2}\| + \cdots + \|x_{n+1} - x_n\|\\ &\leq \left[M^{k-1} + M^{k-2} + \cdots + 1\right] \|x_{n+1} - x_n\|\\ &\leq \frac{1-M^k}{1-M} \|x_{n+1} - x_n\| < \frac{1}{1-M} M^n \|x_1 - x_0\|.\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence and converges to $x^* \in \overline{B(x_0, R)}$.

Finally, we see that x^* is a zero of F. Since

$$\|F(x_n)\| \leq \omega(\eta, \eta)\|x_n - x_{n-1}\|,$$

and $\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$, we obtain $F(x^*) = 0$.

To show the uniqueness, we assume that there exists a second solution $y^* \in \overline{B(x_0, R)}$ and consider the operator $A = [y^*, x^*; F]$. Since $A(y^* - x^*) = F(y^*) - F(x^*)$, if the operator A is invertible then $x^* = y^*$. Indeed,

$$\begin{aligned}\left\|L_0{}^{-1}A - I\right\| &\leq \left\|L_0{}^{-1}\right\| \, \|A - L_0\| \leq \left\|L_0{}^{-1}\right\| \|[y^*, x^*; F] - [x_{-1}, x_0; F]\| \\ &\leq \beta\omega\left(\|y^* - x_{-1}\|, \|x^* - x_0\|\right) \leq \beta\omega\left(\|y^* - x_0\| + \|x_0 - x_{-1}\|, \|x^* - x_0\|\right) \\ &\leq \beta\omega(R + \alpha, R) < 1\end{aligned}$$

and the operator A^{-1} exists. ∎

3. NUMERICAL EXAMPLE

Now we apply the semilocal convergence result given above to the following system:

$$\begin{aligned} x^2 - y + 1 + \frac{1}{9}|x - 1| &= 0, \\ y^2 + x - 7 + \frac{1}{9}|y| &= 0. \end{aligned} \tag{5}$$

We, therefore, have an operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F = (F_1, F_2)$. For $x = (x_1, x_2) \in \mathbb{R}^2$, we take $F_1(x_1, x_2) = x_1{}^2 - x_2 + 1 + (1/9)|x_1 - 1|$, $F_2(x_1, x_2) = x_2{}^2 + x_1 - 7 + (1/9)|x_2|$.

Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $\|x\| = \|x\|_\infty = \max_{1 \leq i \leq 2}|x_i|$. The corresponding norm on $A \in \mathbb{R}^2 \times \mathbb{R}^2$ is

$$\|A\| = \max_{1 \leq i \leq 2} \sum_{j=1}^{2} |a_{ij}|.$$

For $v, w \in \mathbb{R}^2$, we take $[v, w; F] \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ as

$$[v, w; F]_{i1} = \frac{F_i(v_1, w_2) - F_i(w_1, w_2)}{v_1 - w_1}, \quad [v, w; F]_{i2} = \frac{F_i(v_1, v_2) - F_i(v_1, w_2)}{v_2 - w_2}, \qquad i = 1, 2.$$

Therefore,

$$[v, w; F] = \begin{pmatrix} \dfrac{v_1^2 - w_1^2}{v_1 - w_1} & -1 \\ 1 & \dfrac{v_2^2 - w_2^2}{v_2 - w_2} \end{pmatrix} + \frac{1}{9}\begin{pmatrix} \dfrac{|v_1 - 1| - |w_1 - 1|}{v_1 - w_1} & 0 \\ 0 & \dfrac{|v_2| - |w_2|}{v_2 - w_2} \end{pmatrix}$$

and

$$\|[x, y; F] - [v, w; F]\| \leq \|x - v\| + \|y - w\| + \frac{2}{9}.$$

From (b), we consider

$$\omega(u_1, u_2) = u_1 + u_2 + \frac{2}{9}.$$

Now, we apply the Secant method to approximate the solution of $F(x) = 0$.

We choose $z_{-1} = (0.9, 1.1)$ and $z_0 = (1, 1)$. Using iteration (3), after three iterations we obtain

$$z_2 = (1.06867, 2.18207) \qquad \text{and} \qquad z_3 = (1.14038, 2.34476).$$

Then, we take $x_{-1} = z_2$ and $x_0 = z_3$. With the notation of Theorem 2.1, we can easily obtain the following:

$$\alpha = 0.162691, \qquad \beta = 0.479385, \qquad \eta = 0.0199155, \qquad m = 0.194069.$$

In this case, the solution of equation (4) is $R = 0.0263993$. Besides, $\beta\omega(R + \alpha, R) = 0.209832 < 1$ and $M = 0.0245605 < 1$. Therefore, the hypotheses of Theorem 2.1 are fulfilled, what ensures that a unique solution of equation $F(x) = 0$ exists in $\overline{B(x_0, R)}$.

We obtain the vector x^* as the solution of system (5), after nine iterations

$$x^* = (1.15936, 2.36182).$$

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